SPECTRA OF RANDOM NON-ABELIAN REAL-VALUED G-CIRCULANT MATRICES

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Abstract. We analyze the limiting eigenvalue and singular value law for convolution operators on large, not necessarily Abelian, groups in the case of real-valued Gaussian entries, providing an analogue of a result of R. Adamczak, where purely complex-valued entries (i.e. with uncorrelated real and imaginary part) where considered.

It turns out that the spectral behaviour depends on the Frobenius-Schur-indicators of the irreducible representations of the group, and splits up into a real, complex and quaternionic part, which is reflected in the limiting law that is related to the real, complex and quaternionic Plancherel measures of the groups.

We compare this to certain $\mathbb{F}_p^\times$-circulant matrices with deterministic (pseudo-random) entries coming from Number Theory, and find a very different behaviour.

1. Introduction and Preparations

As in [Ada17], let G be a finite group and for a function $X : G \to \mathbb{C}$, consider the convolution operator $P_X : \mathbb{C}^G \to \mathbb{C}^G$ given by

$$(P_X v)(h) := (X * v)(h) := \sum_{g\in G} X_{hg^{-1}} v(g)$$

for $v \in \mathbb{C}^G$ and $h \in G$. In the following, we will mainly consider the case where $X = (X_g)_{g\in G}$ denote i.i.d. real-valued random variables with $\mathbb{E}X_g = 0$ and $\mathbb{E}X_g^2 = 1$, and most importantly, when they are standard real-valued Gaussian random variables.

We recall some standard facts from representation theory of finite groups.

For a finite group G, denote by $\hat{G}$ the collection of irreducible representations of G, which we may assume to be unitary. It is well-known that

$$\sum_{\Lambda\in\hat{G}} (\dim\Lambda)^2 = |G|.$$

We can then define the (normed) measure μ_G on $\mathbb{N}$ given by

$$\mu_G(n) := \mu_G(\{n\}) := \frac{n^2}{|G|} |\{\Lambda \in \hat{G} : \dim\Lambda = n\}|,$$

which we will call the Plancherel measure of G and when convenient, we will view it as a measure on the one-point compactification $\overline{\mathbb{N}} = \mathbb{N} \cup \infty$.

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Next, we define the Frobenius-Schur indicator $\iota(\Lambda)$ of an irreducible representation Λ with character χ by

$$\iota(\Lambda) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2).$$

It is a well-known fact that

$$\iota(\Lambda) = \begin{cases} 1 & \text{if } \Lambda \text{ is real,} \\ 0 & \text{if } \Lambda \text{ is complex,} \\ -1 & \text{if } \Lambda \text{ is quaternionic.} \end{cases}$$

We remark that a real representation is a representation that can be realised over the real numbers, a complex representation is a representation whose character is complex (so that it automatically can only be realised over the complex numbers) and a quaternionic representation is a representation whose character is real but which can only be realised over the complex numbers. Note that a quaternionic representation necessarily has even dimensions and consists of 2×2 blocks of the form

$$\begin{pmatrix} \lambda & \mu \\ -\overline{\mu} & \overline{\lambda} \end{pmatrix} \tag{1}$$

for some $\lambda, \mu \in \mathbb{C}$. We can identify such a matrix with a quaternion $q \in \mathbb{H}$ via $q = \lambda + j\mu$ and remark that matrix multiplication corresponds to quaternion multiplication under this identification.

We can then define the real Plancherel measure $\mu_G^{\mathbb{R}}$ of G by

$$\mu_G^{\mathbb{R}}(n) = \frac{n^2}{|G|}|\{\Lambda \in \hat{G} : \dim\Lambda = n, \iota(\Lambda) = 1\}|$$

and similarly $\mu_G^{\mathbb{C}}$ and $\mu_G^{\mathbb{H}}$.

Since we will view representations over different fields (namely, the real numbers, the complex numbers and the quaternions) depending on context, we introduce the following notation. For clarification, we sometimes write $\dim_{\mathbb{C}}\Lambda$ instead of $\dim\Lambda$ to point out that we view it as a representation over the complex numbers. Then,

- for a real representation, we set $\dim_{\mathbb{R}}\Lambda = \dim_{\mathbb{C}}\Lambda = \dim_{\mathbb{H}}\Lambda$,
- for a complex representation, we set $2\dim_{\mathbb{R}}\Lambda = \dim_{\mathbb{C}}\Lambda = \dim_{\mathbb{H}}\Lambda$ and
- for a quaternionic representation, we set $4\dim_{\mathbb{R}}\Lambda = 2\dim_{\mathbb{C}}\Lambda = \dim_{\mathbb{H}}\Lambda$.

Next, let $\mathcal{N}(\mu,\Sigma)$ denote the Gaussian distribution with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$. In the following, by a real Ginibre ensemble (of dimension d) we mean a random matrix $X \in \mathbb{R}^{d\times d}$ with i.i.d. real-valued entries and distribution $\mathcal{N}(0,\frac{1}{d})$, so that $\sqrt{d}X$ has i.i.d. standard Gaussian entries. A complex Ginibre ensemble (of dimension d) is a random matrix $X \in \mathbb{C}^{d\times d}$ having i.i.d. complex-valued entries with distribution $\mathcal{N}(0,\frac{1}{2d}I_2)$ (viewed as random variables in $\mathbb{R}^2$ by looking at real and imaginary part). A quaternionic Ginibre ensemble (of dimension $2d$) is a random matrix $X \in \mathbb{H}^{d\times d}$ with i.i.d. quaternion-valued entries and distribution $\mathcal{N}(0,\frac{1}{4d}I_4)$. We will in most cases however view it as a random matrix in $\mathbb{C}^{2d\times 2d}$ where in each 2×2 block we identify a quaternion with a 2×2 complex matrix as in (1). This will in particular apply when studiying spectral distributions.

For a matrix $A \in \mathbb{C}^{d\times d}$, we define its eigenvalue measure L_A by

$$L_A := \frac{1}{d}\sum_{i=1}^{d} \delta_{\lambda_i},$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of A counting algebraic multiplicity and where δ_z denotes the Dirac measure on $z \in \mathbb{C}$. We define the singular measure of A by $L_{\sqrt{AA^*}}$, where A^* denotes the conjugate transpose of A. If A is a random matrix, L_A and $L_{\sqrt{AA^*}}$ become random probability measures and we call them the eigenvalue resp. singular value distributions.

We now define $\theta_d^{\mathbb{R}}$ (resp. $\theta_d^{\mathbb{C}}, \theta_d^{\mathbb{H}}$) and $\rho_d^{\mathbb{R}}$ (resp. $\rho_d^{\mathbb{C}}, \rho_d^{\mathbb{H}}$) to be the eigenvalue and singular value distributions of the real (resp. complex, quaternionic) Ginibre ensemble. Here, we view the $2d$-dimensional quaternionic Ginibre ensemble inside $\mathbb{C}^{2d\times 2d}$.

2. Eigenvalue distribution of Gaussian G-circulant matrices

The main goal of this section is to prove the following analogue of [Ada17, Proposition 3.3].

Theorem 1. *Let G be a finite group and let $X = (X_g)_{g\in G}$ be a family of i.i.d. standard real-valued Gaussian random variables. Let also $(\Gamma_\Lambda)_{\Lambda\in\hat{G}}$ be a family of independent Ginibre ensembles, where Γ_Λ is a real (resp. complex, quaternionic) Ginibre ensemble of dimension* $\dim \Lambda$ *when Λ is a real (resp. complex, quaternionic) representation. Then $L_{\frac{1}{\sqrt{|G|}}P_X}$ has the same distribution as the random measure*

$$\sum_{\Lambda\in\hat{G}} \mu_G(\Lambda) L_{\Gamma_\Lambda}$$

and $L_{\frac{1}{\sqrt{|G|}}\sqrt{P_X P_X^*}}$ has the same distribution as the random measure

$$\sum_{\Lambda\in\hat{G}} \mu_G(\Lambda) L_{\sqrt{\Gamma_\Lambda \Gamma_\Lambda^*}}.$$

The main tool we will use to prove this Theorem is an analogue of [Ada17, Lemma 3.5].

Lemma 2. *Let $X = (X_g)_{g\in G}$ be independent real-valued random variables such that for each $g \in G$, $\mathbb{E}X_g = 0$ and $\mathbb{E}X_g^2 = 1$. Consider the $|G|$ random variables consisting of*

- *$\hat{X}(\Lambda)_{ij}$, $i, j = 1, \dots, \dim_{\mathbb{R}} \Lambda$ for real irreducible representations Λ of G,*
- *$\mathcal{R}\hat{X}(\Lambda)_{ij}, \mathcal{I}\hat{X}(\Lambda)_{ij}$, $i, j = 1, \dots, \dim_{\mathbb{C}} \Lambda$, ranging over the complex irreducible representations of G, but out of an irreducible representation and its complex conjugate we pick only precisely one. Note that the complex conjugate of a complex irreducible representation is again a complex irreducible representation and moreover non-equivalent to the initital representation;*
- *$\mathcal{R}\hat{X}(\Lambda)_{ij}, \mathcal{I}\hat{X}(\Lambda)_{ij}, \mathcal{J}\hat{X}(\Lambda)_{ij}, \mathcal{K}\hat{X}(\Lambda)_{ij}$, $i, j = 1, \dots, \dim_{\mathbb{H}} \Lambda$ ranging over all quaternionic representations Λ viewed as matrices with quaternionic entries in the usual way. This is equivalent to taking the real and imaginary parts of the entries $\hat{X}(\Lambda)_{ij}$ of all quaternionic representations viewed as complex matrices, where i ranges only over odd indices.*

These random variables are pairwise uncorrelated, and if $\hat{X}$ denotes an arbitrary random variable of this collection, associated to a representation Λ, then we have

$$\mathbb{E}\hat{X} = 0, \qquad \mathbb{E}\hat{X}^2 = \frac{|G|}{\dim_{\mathbb{R}} \Lambda}.$$

Note that $\dim_{\mathbb{R}} \Lambda$ can be interpreted as the dimension of Λ over the real numbers in a quite rigorous sense. More prescisely, it is well-known (see [JL01, chapter 23], in particular Proposition 23.6) that

- an irreducible representation over $\mathbb{C}$ with Frobenius-Schur indicator 1 stays irreducible over $\mathbb{R}$,
- an irreducible representation Λ over $\mathbb{C}$ of dimension d with Frobenius-Schur indicator 0 can be naturally identified with an irreducible representation over $\mathbb{C}$ of dimension $2d$. Two representations Λ_1, Λ_2 identify with the same representation iff they are either equal or conjugate (up to equivalence);
- an irreducible representation over $\mathbb{C}$ of (complex) dimension $2d$ with Frobenius-Schur indicator -1 can be naturally identified with an irreducible representation over $\mathbb{R}$ of dimension $4d$. The resulting representations coincide iff the initial representations coincide (up to equivalence).

The proof of Lemma 2 makes heavy use of what is sometimes referred to as the Great Orthogonality Theorem. A reference for this is [Dia88, Chapter 2B, Corollary 2,3].

Lemma 3. *Let G be a finite group, and let $\Lambda_1, \dots, \Lambda_r$ be a complete list of its irreducible representations, which we may further assume to be unitary. Then for any $m, n = 1, \dots, r$ and indices $i, j = 1, \dots, \dim \Lambda_m$ as well as $k, l = 1, \dots, \dim \Lambda_n$, we have*

$$\sum_{g \in G} \Lambda_m(g)_{ij} \overline{\Lambda_n(g)_{kl}} = \frac{|G|}{\dim \Lambda_m} \delta_{mn} \delta_{ik} \delta_{jl}. \tag{2}$$

Note that for the right-hand side to be non-zero we must have $\dim \Lambda_m = \dim \Lambda_n$, so the statement is symmetric.

Proof of Lemma 2. Firstly, it is immediate from the definitions that all the random variables have zero expectation.

Let Λ_1, Λ_2 be any irreducible (unitary) representations of G. Noting that

$$\mathcal{R}z\mathcal{R}w = \frac{1}{2}\mathcal{R}[(z + \overline{z})w]$$

for $z, w \in \mathbb{C}$, we obtain

$$\begin{aligned} \mathbb{E}\left[\mathcal{R}\hat{X}(\Lambda_1)_{ij}\mathcal{R}\hat{X}(\Lambda_2)_{kl}\right] &= \frac{1}{2}\mathcal{R}\mathbb{E}\left[\left(\sum_{g \in G} X_g \left(\Lambda_1(g)_{ij} + \overline{\Lambda_1(g)_{ij}}\right)\right)\left(\sum_{g \in G} X_g \Lambda_2(g)_{kl}\right)\right] \\ &= \frac{|G|}{2\dim \Lambda_1} \delta_{\Lambda_1 \Lambda_2} \delta_{ik} \delta_{jl} + \frac{1}{2} \sum_{g \in G} \Lambda_1(g)_{ij} \Lambda_2(g)_{kl}, \end{aligned} \tag{3}$$

where in the last step we have applied Lemma 3, and where $\delta_{\Lambda_1\Lambda_2}$ is 1 if the representations are equal (as opposed to equivalent) and 0 if they are non-equivalent. Similar arguments imply

$$\mathbb{E}\left[\mathcal{R}\hat{X}(\Lambda_1)_{ij}\mathcal{I}\hat{X}(\Lambda_2)_{kl}\right] = 0 \tag{4}$$

and

$$\mathbb{E}\left[\mathcal{I}\hat{X}(\Lambda_1)_{ij}\mathcal{I}\hat{X}(\Lambda_2)_{kl}\right] = \frac{|G|}{2\dim\Lambda}\delta_{\Lambda_1\Lambda_2}\delta_{ik}\delta_{jl} - \frac{1}{2}\sum_{g\in G}\Lambda_1(g)_{ij}\Lambda_2(g)_{kl}. \tag{5}$$

Note that we can not infer anything from Lemma 3 when the representations are equivalent, but not equal; hence, we can not directly apply them to Λ_1 and $\overline{\Lambda_2}$ to get an analogous expression for the second part (at least not in the quaternionic case).

Equations (3), (4) and (5) in fact imply the claim. Suppose first that Λ_1 and Λ_2 are irreducible unitary representations such that neither Λ_1 and Λ_2 nor Λ_1 and $\overline{\Lambda_2}$ are equivalent. Then we may invoke Lemma 3, applied to Λ_1 and $\overline{\Lambda_2}$ to deduce that any two random variables that are associated to the respective representations must be uncorrelated. This settles in particular the case when the representations have different Frobenius-Schur indicators.

Now, suppose that both representations are real. If they are not equal then we may assume them to be non-equivalent so that automatically, Λ_1 and $\overline{\Lambda_2}$ are also non-equivalent and we land in the first case. Thus, suppose that $\Lambda := \Lambda_1 = \Lambda_2 = \overline{\Lambda_2}$, so that (3) together with Lemma 3 implies

$$\mathbb{E}\left[\hat{X}(\Lambda)_{ij}\hat{X}(\Lambda)_{kl}\right] = \frac{|G|}{\dim\Lambda}\delta_{ik}\delta_{jl},$$

which settles this case.

Next, let Λ_1 and Λ_2 be complex representations. Note that a complex representation is not equivalent to its complex conjugate, so that we may again apply Lemma 3, which gives us

$$\sum_{g\in G}\Lambda_1(g)_{ij}\Lambda_2(g)_{kl} = \frac{|G|}{\dim\Lambda_1}\delta_{\Lambda_1\overline{\Lambda_2}}\delta_{ik}\delta_{jl}.$$

But from our choice of random variables we may assume Λ_1 and $\overline{\Lambda_2}$ to be non-equivalent, so that the latter expression evaluates to 0. One verifies that this together with (3), (4) and (5) implies this part of the claim.

Lastly, suppose that Λ_1 and Λ_2 are quaternionic representations. Since a quaternionic representation is equivalent, but not equal to its complex conjugate, we may assume $\Lambda := \Lambda_1 = \Lambda_2$, but we can not directly apply Lemma 3. However, when viewed as matrices with complex entries, we know that Λ consists of 2×2 blocks of the form (1), so that we have

$$\Lambda(g)_{kl} = \pm\overline{\Lambda(g)_{k+1,l\pm 1}}$$

depending on whether l is odd or even (and making use of the fact that we may assume k to be odd). Hence, we infer

$$\sum_{g\in G}\Lambda_1(g)_{ij}\Lambda_2(g)_{kl} = \pm\sum_{g\in G}\Lambda_1(g)_{ij}\overline{\Lambda_2(g)_{k+1,l\pm 1}} = 0$$

by Lemma 3, noting that i and $k+1$ can never coincide because the first one is odd and the second one is even. One verifies that this concludes the proof. □

Proof of Theorem 1. We have already verified in Lemma 2 that the random variables of interest (listed in the Lemma) are pairwise uncorrelated and have the correct expectation and variance. It remains to show their joint independence in view of [Ada17, Corollary 3.2]. But we are now in the Gaussian case, and here it suffices to show that the random variables listed in Lemma 2 are jointly Gaussian to deduce the joint independence (from pairwise uncorrelatedness). But this is clear since the collection is just a linear transformation of the (jointly Gaussian) random variables $(X_g)_{g\in G}$, and the determinant of the transformation matrix A is given by

$$\prod_{\Lambda\in\hat{G}}\left(\frac{|G|}{\dim_{\mathbb{R}}\Lambda}\right)^{(\dim_{\mathbb{C}}\Lambda)^2}.$$

This can be verified quickly by checking that $\sqrt{AA^*}$ is a diagonal matrix with the corresponding values on the diagonal (as a consequence of Lemma 2). □

3. Real Eigenvalues

Note that the complex as well as the quaternionic Ginibre ensemble almost surely possess no real eigenvalues, while for the real Ginibre ensemble, we have the following fact from [EKS94].

Theorem 4. *Let E_n denote the expected number of real eigenvalues of the real $n\times n$ Ginibre ensemble. Then we have*

$$\lim_{n\to\infty}\frac{E_n}{\sqrt{n}}=\sqrt{\frac{2}{\pi}}.$$

More precisely, if n is even,

$$E_n=\sqrt{2}\sum_{k=0}^{n/2-1}\frac{(4k-1)!!}{(4k)!!}$$

while for n odd,

$$E_n=1+\sqrt{2}\sum_{k=0}^{(n-1)/2-1}\frac{(4k-3)!!}{(4k-2)!!}.$$

Hence, we obtain the following

Corollary 5. *Let G be a finite group, let $(X_g)_{g\in G}$ be a family of i.i.d. standard Gaussian random variables and let E denote the expected number of real eigenvalues of $\frac{1}{\sqrt{|G|}}P_X$. Then we have*

$$E=\sum_{\substack{\Lambda\in\hat{G}\\ \iota(\Lambda)=1}}(\dim\Lambda)E_{\dim\Lambda}.$$

Proof. Invoking Theorem 1, we have already remarked that only the real Ginibre ensembles contribute to real eigenvalues. Now [Ada17, Proposition 3.1] tells us that the eigenvalues of each Γ_Λ appear in $\frac{1}{\sqrt{|G|}}P_X$ with multiplicity $\dim\Lambda$. This gives the claim. □

There are many examples of sequences of finite groups where we can study the asymptotic behaviour of the expected number of real eigenvalues.

Suppose for example that $G_N = S_N$ are the symmetric groups. The irreducible representations of the symmetric groups are all real and their dimensions correspond to the hook numbers h_λ associated to the partitions λ of N. Denoting the expected number of real eigenvalues of $\frac{1}{\sqrt{|S_N|}} P_X$ by E^N, we have

$$E^N = \sum_{\lambda \dashv N} h_\lambda E_{h_\lambda} \sim \sum_{\lambda \dashv N} h_\lambda^{3/2}$$

using the first part of Theorem 4. An application of [VK85, Theorem 1] quickly yields the existence of an explicit constant $c > 0$ such that

$$e^{-c\sqrt{N}} (N!)^{3/4} \ll E^N \ll e^{c\sqrt{N}} (N!)^{3/4}$$

(c = 3 works). In particular, the proportion of real eigenvalues decays essentially like $|G_N|^{-1/4}$ (and since we will later prove convergence of the eigenvalue distribution to the circular law, it has to decay).

Next, let $G_N = Dih_N$ be the dihedral groups (so that G_N has $2N$ elements). All irreducible representations are real, of which $O(1)$ have dimension 1 and $N/2 + O(1)$ have dimension 2 (one could easily be more precise here). With the notation from before, we obtain

$$E^N = 2\sqrt{2}\frac{N}{2} + O(1) = \sqrt{2}N + O(1).$$

Thus, the proportion of real eigenvalues converges to $\sqrt{2}/2$.

4. Limiting Eigenvalue and Singular Value Distribution

We have the following analogue of [Ada17, Theorem 1.10].

Theorem 6. *Let G_N be a sequence of finite groups with $|G_N| \to \infty$. Assume that the real, complex and quaternionic Plancherel measures $\mu_N^{\mathbb{R}}, \mu_N^{\mathbb{C}}$ and $\mu_N^{\mathbb{H}}$ converge weakly to measures $\mu^{\mathbb{R}}, \mu^{\mathbb{C}}$ and $\mu^{\mathbb{H}}$ on $\overline{\mathbb{N}}$ such that their sum defines a probability measure. For each N, let $X^N = (X_g^N)_{g \in G}$ be i.i.d. standard real Gaussian random variables. Then the empirical spectral measure L_N^e of the matrix $\frac{1}{\sqrt{|G_N|}} P_{X^N}$ converges weakly in probability to the deterministic measure L_∞^e on $\mathbb{C}$ with density*

$$\frac{dL_\infty^e(z)}{dz} = \sum_{n \in \overline{\mathbb{N}}} \left(\mu^{\mathbb{R}}(n) \frac{d\theta_n^{\mathbb{R}}(z)}{dz} + \mu^{\mathbb{C}}(n) \frac{d\theta_n^{\mathbb{C}}(z)}{dz} + \mu^{\mathbb{H}}(n) \frac{d\theta_n^{\mathbb{H}}(z)}{dz} \right).$$

Proof. The proof proceeds in almost exactly the same way as the proof of [Ada17, Theorem 1.10]. The only difference is that one has to treat all three cases, and then triangle inequality gives the claim. ☐

Next, we claim an analogue of [Ada17, Theorem 1.5].

Theorem 7. *Let G_N be a sequence of finite groups with $|G_N| \to \infty$, and suppose that the sequences of Plancherel measures $\mu_N^{\mathbb{R}}, \mu_N^{\mathbb{C}}$ and $\mu_N^{\mathbb{H}}$ converge weakly to measure $\mu^{\mathbb{R}}, \mu^{\mathbb{C}}$ and $\mu^{\mathbb{H}}$ on $\overline{\mathbb{N}}$ such that their sum defines a probability measure. For each N, let $X^N = (X_g^N)_{g \in G_N}$ be i.i.d. copies of a* *real-valued random variable* ξ *with* $\mathbb{E}\xi = 0$ *and* $\mathbb{E}\xi^2 = 1$. *Then the singular value distributions of* $\frac{1}{\sqrt{|G_N|}} P_{X^N}$ *converge weakly in probability to the deterministic measure* L^s_∞ *on* $\mathbb{R}^+$ *with density*

$$\frac{dL^s_\infty(x)}{dx} = \sum_{n\in\overline{\mathbb{N}}} \mu^{\mathbb{R}}(n)\frac{d\rho^{\mathbb{R}}_n(x)}{dx} + \sum_{n\in\overline{\mathbb{N}}} \mu^{\mathbb{C}}(n)\frac{d\rho^{\mathbb{C}}_n(x)}{dx} + \sum_{n\in\overline{\mathbb{N}}} \mu^{\mathbb{H}}(n)\frac{d\rho^{\mathbb{H}}_n(x)}{dx}. \tag{6}$$

Proof. The proof of this statement also proceeds in a very similar way as in the complex case. While the real representation theory is slightly more difficult than the complex one, there are some tiny simplifications in this part of the proof, essentially because we do not have to consider ξ over $\mathbb{R}^2$, so that objects such as its covariance matrix become simpler. On the other hand, because the random variables in Lemma 2 have a more complicated structure, we have to be a little more careful in some details

We will not go through every part of the proof step by step, but will highlight the changes that have to be made in order to adapt it to this case. We only give a complete proof of the following Lemma, which contains the most significant changes.

Lemma 8. *Fix a positive integer* n. *Let* G_N *be a sequence of finite groups with* $|G_N| \to \infty$, *and let* Λ_N, Δ_N *be two irreducible unitary representations of* G_N *of dimension at most* n, *neither equal nor complex conjugates of each other. Let also* $f : \mathbb{R}_+ \to \mathbb{R}$ *be a bounded continuous function. Then as* $N \to \infty$ *we have the convergence*

$$\left| \mathbb{E}\int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Lambda_N)\hat{X}^N(\Lambda_N)^*}} - \mathbb{E}\int_{\mathbb{R}_+} f dL_{\sqrt{\Gamma_{\Lambda_N}\Gamma^*_{\Lambda_N}}} \right| \to 0 \tag{7}$$

and

$$Cov\left(\int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Lambda_N)\hat{X}^N(\Lambda_N)^*}}, \int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Delta_N)\hat{X}^N(\Delta_N)^*}}\right) \to 0. \tag{8}$$

Proof. Since n is fixed, by splitting G_N into a finite number of subsequences, we can assume that $\dim\Lambda_N = k$, $\dim\Delta_N = l$ and that both representations have constant Frobenius-Schur indicators. Consider the couple of random matrices

$$Z_N := \left(\frac{1}{\sqrt{|G_N|}}\hat{X}^N(\Lambda_N), \frac{1}{\sqrt{|G_N|}}\hat{X}^N(\Delta_N)\right)$$

as a random vector in $\mathbb{R}^{\beta(\Lambda_N)k^2+\beta(\Delta_N)l^2}$, where $\beta(\Lambda) = 1$ when Λ is real, and $\beta(\Lambda) = 2$ when Λ is complex or quaternionic. The components of Z_N listed in Lemma 2 corresponding to Λ_N and Δ_N are uncorrelated and have variance $\frac{1}{\dim_{\mathbb{R}}(\Lambda_N)}$ and $\frac{1}{\dim_{\mathbb{R}}(\Delta_N)}$ (by the Lemma). Again, since $\Lambda_N(g)$ and $\Delta_N(g)$ are unitary, they are bounded independently of N and the Lindeberg condition for the corresponding components of $\frac{1}{\sqrt{|G_N|}}\sum_{g\in G_N} X^N_g(\Lambda_N(g), \Delta_N(g))$ is trivially satisfied. Hence, these components converge in distribution to the same components of $(\Gamma^{(1)}, \Gamma^{(2)})$, where $\Gamma^{(i)}$ are independent Ginibre ensembles of the same size and type as Λ_N resp. Δ_N (viewed as real random vectors). However, this implies the convergence in distribution of Z_N to $(\Gamma^{(1)}, \Gamma^{(2)})$: In the real and complex case, this is a void statement because in Lemma 2 we take all components; in the quaternionic case, this is essentially claiming that if Y_N is a random complex vector such that $Y_N \to Y$ in distribution then $(Y_N, \pm\overline{Y_N}) \to (Y, \pm\overline{Y})$ in distribution (where the sign is independent of N), which is an immediate consequence of the Cramér-Wold Theorem. Also note that $(\Gamma^{(1)}, \Gamma^{(2)})$ has the same distribution as $(\Gamma_{\Lambda_N}, \Gamma_{\Delta_N})$. Now, consider the map

$$A \mapsto \int_{\mathbb{R}^+} f dL_{\sqrt{AA^*}}.$$

For fixed f as above, this is also bounded continuous (bounded by $\|f\|_\infty$ and continuous in the entries of A), and hence we obtain (7) by applying the convergence in distribution to $\frac{1}{\sqrt{|G_N|}}\hat{X}^N(\Lambda_N)$. The remainder proceeds in exactly the same way as in [Ada17, Lemma 4.1], noting that the map

$$(A, B) \mapsto \int_{\mathbb{R}^+} f dL_{\sqrt{AA^*}} \int_{\mathbb{R}^+} f dL_{\sqrt{BB^*}}$$

is also bounded continuous and therefore we have

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Lambda_N)\hat{X}^N(\Lambda_N)^*}} \int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Delta_N)\hat{X}^N(\Delta_N)^*}}\right] \\ \to &\mathbb{E}\left[\int_{\mathbb{R}^+} f dL_{\sqrt{\Gamma^{(1)}(\Gamma^{(1)})^*}} \int_{\mathbb{R}^+} f dL_{\sqrt{\Gamma^{(2)}(\Gamma^{(2)})^*}}\right] = \mathbb{E}\int_{\mathbb{R}^+} f dL_{\sqrt{\Gamma^{(1)}(\Gamma^{(1)})^*}} \mathbb{E}\int_{\mathbb{R}^+} f dL_{\sqrt{\Gamma^{(2)}(\Gamma^{(2)})^*}}, \end{aligned}$$

which gives (8). □

Next, [Ada17, Lemma 4.2] proceeds in almost exactly the same way, the only difference is that the random variables $(Y_g^N)_{g\in G_N}$ should of course also be real-valued i.i.d. standard Gaussian (and thus we do not need to conjugate X_g^N or Y_g^N).

Similarly, the concentration of measure argument works without changes and we obtain [Ada17, Proposition 4.6]:

Proposition 9. *In the situation of Theorem 7, assume additionally that ξ is bounded. Let Λ_N be a sequence of irreducible representations of G_N with $d_N := \dim \Lambda_N \to \infty$ as $N \to \infty$. Then for any positive integer k we have*

$$\frac{1}{d_N} tr\left(\frac{1}{|G_N|}\widehat{X^N}(\Lambda_N)\widehat{X^N}(\Lambda_N)^*\right)^k \to \int_0^\infty x^{2k} d\rho_\infty(x).$$

In particular, $L_{\frac{1}{\sqrt{|G_N|}}\sqrt{\widehat{X^N}(\Lambda_N)\widehat{X^N}(\Lambda_N)^*}}$ converges weakly in probability to ρ_∞.

Note that even though the statement is precisely the same, there is slightly more behind this version. The important point is that, as usual, Λ_N can have one of the three diffferent types, but we have $\rho_\infty := \rho_\infty^{\mathbb{C}} = \rho_\infty^{\mathbb{R}} = \rho_\infty^{\mathbb{H}}$. We can thus subpartition the representations into those having constant type, and then the result follows by noting that

$$\frac{1}{n}\mathbb{E}\,\mathrm{tr}(\Gamma_n\Gamma_n^*)^k \to \int_0^\infty x^{2k} d\rho_\infty(x)$$

as $n \to \infty$, where Γ_n denotes an n-dimensional Ginibre ensemble of any (say fixed) type.

Removing the boundedness assumption of ξ to obtain [Ada17, Proposition 4.7] also goes through without difficulty, in fact it simplifies because the covariance matrix of ξ reduces to a scalar given by the variance of ξ.

Lastly, the conclusion of the proof of Theorem 7 is a fairly straight-forward analogue of the proof for [Ada17, Theorem 1.5]. One additionally needs to split up the error terms into parts corresponding to real, complex and quaternionic Plancherel measures as before, using triangle inequality.

The other difference is that in Lemma 8, we had the additional assumption that the representations are not conjugates of each other (which is only relevant in the complex case). Thus, we only get the seemingly weaker estimate

$$\max_{\substack{\Lambda\neq\Delta,\overline{\Delta}\in\hat{G}_N\\ \dim\Lambda,\dim\Delta\leq n_0}} \operatorname{Cov}\left(\int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Lambda)\hat{X}^N(\Lambda)^*}}, \int_{\mathbb{R}_+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Delta)\hat{X}^N(\Delta)^*}}\right) \to 0$$

as $N\to\infty$ instead of [Ada17, (4.12)]. However, for any representation Λ the measures $L_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Lambda)\hat{X}^N(\Lambda)^*}}$ and $L_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\overline{\Lambda})\hat{X}^N(\overline{\Lambda})^*}}$ coincide, so that it nonetheless suffices to show that

$$\operatorname{Var}\Bigg(\sum_{\substack{\Lambda\in\hat{G}_N\\ \dim\Lambda\leq n_0}} \mu_{G_N}(\Lambda)\int_{\mathbb{R}^+} f dL_{\frac{1}{\sqrt{|G_N|}}\sqrt{\hat{X}^N(\Lambda)\hat{X}^N(\Lambda)^*}}\Bigg) \to 0,$$

as is proved there. Hence, we obtain the claim.

□

5. Eigenvalue Distribution of a Pseudorandom G-Circulant

Let p denote a prime, $\mathbb{F}_p$ the field of cardinality p and $\mathbb{F}_p^\times$ its multiplicative group. For $a,b\in\mathbb{F}_p^\times$, we define the Kloosterman sum

$$K_p(a,b) := \frac{1}{\sqrt{p}}\sum_{x\in\mathbb{F}_p^\times} e\left(\frac{ax+b\overline{x}}{p}\right),$$

where $\overline{x}$ denotes the inverse of x in $\mathbb{F}_p^\times$ and $e(z) := e^{2\pi i z}$. It is an elementary verification that $K_p(a,b)\in\mathbb{R}$. While of course being a deterministic object, this has pseudorandom properties in the following sense (see [Kat88, 13.5.3]).

Theorem 10. *Consider the sequence of random variables $(K_p)_{p \text{ prime}}$, given by*

$$(a,b)\mapsto K_p(a,b)$$

on the finite probability space $\mathbb{F}_p^\times\times\mathbb{F}_p^\times$ with the uniform probability measure. Then as $p\to\infty$, we have convergence in distribution of $K_p\xrightarrow{d}K$, where K is a random variable distributed according to the Sato-Tate law μ_{ST}, given by the density

$$\frac{d\mu_{ST}(x)}{dx} = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}$$

for $x\in[-2,2]$.

Next, define what we call the circulant Kloosterman operator, given by the matrix $\tilde{K}_p := (\frac{1}{\sqrt{p}}K_p(\overline{a},b))_{a,b\in\mathbb{F}_p^\times}$. One quickly verifies that this is indeed a circulant matrix for the group $\mathbb{F}_p^\times$. It has deterministic entries for every p, but they are pseudorandom and Sato-Tate distributed (before the additional normalization with $1/\sqrt{p}$) in the above sense. Thus, it is an interesting question how the eigenvalue and singular value distributions of $\tilde{K}_p$ look like; in fact, this is not a very hard problem:

Proposition 11. *As $p \to \infty$, the eigenvalue distribution of $\tilde{K}_p$ converges weakly to the uniform distribution on the complex unit circle.*

Proof. Let χ be a character of $\mathbb{F}_p^\times$, i.e. a homomorphism $\mathbb{F}_p^\times \to \mathbb{C}^\times$. Note that this is the same as an irreducible representation of $\mathbb{F}_p^\times$, and in particular (since the group is Abelian), there are $p-1$ characters. If $\chi = \chi_0 \equiv 1$ is the trivial character, it is immediate that

$$\tilde{K}_p\chi_0 = \frac{1}{p}\chi_0,$$

so that χ_0 is an eigenfunction of $\tilde{K}_p$ with eigenvalue $\frac{1}{p}$. Now, suppose that χ is a non-trivial character, recall that the Gauss sum $\tau(\chi)$ associated to a character χ is given by

$$\tau(\chi) = \sum_{a\in\mathbb{F}_p^\times} \chi(a)e\left(\frac{a}{p}\right)$$

and that for a non-trivial character we have $|\tau(\chi)| = \sqrt{p}$. Moreover, for such a character and any $n \in \mathbb{F}_p^\times$, we have the equality

$$\sum_{a\in\mathbb{F}_p^\times} \chi(a)e\left(\frac{an}{p}\right) = \overline{\chi(n)}\tau(\chi).$$

Noting that $\chi(\overline{m}) = \overline{\chi(m)}$, we obtain

$$\begin{aligned}(\tilde{K}_p\chi)(n) &= \frac{1}{\sqrt{p}}\sum_{m\in\mathbb{F}_p^\times} K_p(\overline{n},m)\chi(m) = \frac{1}{p}\sum_{x\in\mathbb{F}_p^\times} e\left(\frac{\overline{n}x}{p}\right)\sum_{m\in\mathbb{F}_p^\times} e\left(\frac{m\overline{x}}{p}\right)\chi(m)\\ &= \frac{\tau(\chi)}{p}\sum_{x\in\mathbb{F}_p^\times} e\left(\frac{\overline{n}x}{p}\right)\chi(x) = \frac{\tau(\chi)^2}{p}\chi(n).\end{aligned}$$

Hence, any non-trivial character χ is also an eigenfunction of $\tilde{K}_p$ with eigenvalue $\frac{\tau(\chi)^2}{p}$. We have already remarked that these quantities have norm one, and moreover they equidistribute on the complex unit circle by [Kat88, 9.3]. Since the characters form an orthonormal basis of $\mathbb{C}^{p-1}$, this gives the claim. ☐

Note that the normalization is the same as the usual one by $\frac{1}{\sqrt{\dim}}$ up to a negligent factor, but we have chosen $\frac{1}{\sqrt{p}}$ because it is more convenient for the computation.

What is remarkable about this Proposition is that the behaviour of the eigenvalues in the pseudorandom case here is entirely different from the one for $\mathbb{F}_p^\times$-circulant matrices with independent

Sato-Tate distributed random variables. In that case, [Mec12, Theorem 4.1] tells us that the limiting eigenvalue distribution is given by a complex standard Gaussian, because almost all characters are complex.

We can also consider the Birch sums

$$B_p(a) := \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p} e\left(\frac{ax + x^3}{p}\right)$$

and from this define what we will call the circulant Birch operator, $\tilde{B}_p := \left(\frac{1}{\sqrt{p}} B_p(\overline{a}b)\right)_{a,b \in \mathbb{F}_p^\times}$. The Birch sums satisfiy a very similar pseudorandomness property to the Kloosterman sums:

Theorem 12. *For p prime, consider the random variable $a \mapsto B_p(a)$ on the finite probability space $\mathbb{F}_p^\times$ with the uniform probability measure. Then as $p \to \infty$, we have convergence in distribution $B_p \to B$, where B is again distributed according to the Sato-Tate law.*

However, the eigenvalues of $\tilde{B}_p$ satisfy a slightly different law than for $\tilde{K}_p$:

Proposition 13. *The eigenvalue distribution of $\tilde{B}_p$ satisfies the following properties.*

(i) In the limit $p \to \infty$ ranging over $p \equiv 2$ (3), the eigenvalue distribution of $\tilde{B}_p$ converges weakly to the uniform distribution $\mathcal{U}(S^1)$ on the complex unit circle.

(ii) For $p \to \infty$ ranging over $p \equiv 1$ (3), the eigenvalue distribution of $\tilde{B}_p$ converges weakly to

$$\frac{2}{3}\delta_0 + \frac{1}{3}\mathcal{U}(S^1) * \mathcal{U}(S^1) * \mathcal{U}(S^1),$$

where δ_0 denotes the Dirac measure at 0 and $*$ is the convolution of probability measures corresponding to the distribution of the sum of independent random variables.

Proof. Let χ be a multiplicative character. Then we have

$$\begin{aligned}(\tilde{B}_p\chi)(n) &= \frac{1}{\sqrt{p}} \sum_{m \in \mathbb{F}_p^\times} B_p(\overline{n}m)\chi(m) = \frac{1}{p} \sum_{m \in \mathbb{F}_p^\times} \sum_{x \in \mathbb{F}_p} e\left(\frac{\overline{n}mx + x^3}{p}\right)\chi(m) \\ &= \frac{1}{p} \sum_{x \in \mathbb{F}_p} e\left(\frac{x^3}{p}\right) \sum_{m \in \mathbb{F}_p^\times} e\left(\frac{\overline{n}mx}{p}\right)\chi(m) = \frac{\chi(n)\tau(\chi)}{p} \sum_{x \in \mathbb{F}_p} e\left(\frac{x^3}{p}\right)\overline{\chi(x)}.\end{aligned}$$

Hence, any multiplicative character χ is an eigenfunction of $\tilde{B}_p$ with eigenvalue

$$\lambda_\chi = \frac{\tau(\chi)}{p} \sum_{x \in \mathbb{F}_p} e\left(\frac{x^3}{p}\right)\overline{\chi(x)}.$$

Now, suppose first that $p \equiv 2$ (3). Then $x \mapsto x^3$ defines a bijection on $\mathbb{F}_p$ (and on $\mathbb{F}_p^\times$). Thus, we have an inverse map denoted by $x \mapsto x^{1/3}$, and the same holds for the character group since it is (non-canonically) isomorphic to $\mathbb{F}_p^\times$. Thus, we can write

$$\lambda_\chi = \frac{\tau(\chi)}{p} \sum_{x \in \mathbb{F}_p} e\left(\frac{x}{p}\right)\overline{\chi(x^{1/3})} = \frac{\tau(\chi)\tau(\overline{\chi^{1/3}})}{p}.$$

Next, we claim that in the case $p \equiv 1$ (3), the eigenvalue λ_χ vanishes when χ is not the cube of another character (which holds for two thirds of the characters). To see this, let χ be non-trivial and note that we have

$$\lambda_\chi = \frac{\tau(\chi)}{p} \sum_{x \in \mathbb{F}_p} e\left(\frac{x^3}{p}\right) \overline{\chi(x)} = \frac{\tau(\chi)}{p} \sum_{y,z \in \mathbb{F}_p} e\left(\frac{y}{p}\right) \overline{\chi(z)} \frac{1}{p-1} \sum_{\chi'} \chi'(y\overline{z}^3)$$
$$= \frac{\tau(\chi)}{p(p-1)} \sum_{\chi'} \tau(\chi') \sum_{z \in \mathbb{F}_p} \overline{\chi\chi'^3}(z).$$

But the inner sum vanishes when $\chi\chi'^3$ is non-trivial, hence the claim follows (note that χ is a cube iff $\overline{\chi}$ is). □

6. Examples

We start by studying the case where G_N is a sequence of Abelian groups such that $|G_N| \to \infty$ to recover the corresponding result in [Mec12]. Note that all irreducible representations of an abelian group are one-dimensional and thus either real or complex (but not quaternionic), so that $\mu_N^{\mathbb{R}}(1)$ is the proportion of real characters of G_N, $\mu_N^{\mathbb{C}}(1)$ is the proportion of complex characters of G_N and the assumption that the Plancherel measures should converge weakly transforms to the assumption that the proportion of real characters p_N of G_N should converge to some value p. Since $\theta_1^{\mathbb{R}}$ is the standard real Gaussian distribution with measure denoted by $\gamma^{\mathbb{R}}$, and $\theta_1^{\mathbb{C}}$ is the standard complex Gaussian distribution with measure denoted by $\gamma^{\mathbb{C}}$, we obtain the limiting law

$$(1-p)\gamma^{\mathbb{C}} + p\gamma^{\mathbb{R}}$$

as in [Mec12, Theorem 4.1].

Now, suppose that $G_N = S_N$ are the symmetric groups. We have already noted that all representations are real, and it is well-known that $\mu_N^{\mathbb{R}}$ converges weakly to the Dirac measure at ∞. Thus, we obtain the limiting eigenvalue distribution with density $\theta_\infty^{\mathbb{R}}$, which is the circular law.

Next, let $G_q = GL(2,q)$, indexed by prime powers. It is well-known that there are precisely $q^2 - 1$ irreducible representations, and one verifies that none of them are quaternionic, $O(q)$ are real and the rest are complex. Moreover, there are $O(q)$ representations of dimension 1 while the rest has dimension at least $q-1$. Hence, $\mu_N^{\mathbb{C}}$ converges weakly to the Dirac measure at infinity, so that we obtain the limiting eigenvalue distribution with density $\theta_\infty^{\mathbb{C}}$, which is again the circular law.

Let Q be the quaternion group, and let $G_N = Q \times (\mathbb{Z}/N\mathbb{Z})$. The irreducible representations of a product of groups coincide with the tensor products of irreducible representations of the groups, and the characters of the product are the products of the characters. Now the characters of Q are real, while almost all characters of $\mathbb{Z}/N\mathbb{Z}$ are complex, so that the characters of the product are almost all complex. Moreover, we have $\mu_Q^{\mathbb{R}}(1) = \mu_Q^{\mathbb{H}}(2) = \frac{1}{2}$ and thus we obtain the limiting spectral distribution

$$L_\infty^e = \frac{1}{2}\theta_1^{\mathbb{C}} + \frac{1}{2}\theta_2^{\mathbb{C}}.$$

Consider this time the sequence of groups $G_N = Q \times (\mathbb{Z}/2\mathbb{Z})^N$. Note that if G, H are finite groups and Λ_1, Λ_2 are irreducible representations of G resp. H, then we have

$$\iota(\Lambda_1 \otimes \Lambda_2) = \iota(\Lambda_1)\iota(\Lambda_2),$$

and one quickly infers the limiting eigenvalue distribution

$$L^e_\infty = \frac{1}{2}\theta_1^{\mathbb{R}} + \frac{1}{2}\theta_2^{\mathbb{H}}.$$

Lastly, let G be any non-Abelian group, and set $G_N := G^N$. One verifies easily that the Plancherel measure of G_N converges weakly to the Dirac measure at infinity. Moreover, it is not hard to see that the real, complex and quaternionic Plancherel measure all weakly converge (even though we do not believe this to be necessary when the limit measure is the Dirac measure at infinity). Since the limiting eigenvalue distributions of all Ginibre ensemble converge to the circular laws as the dimension increases, the limiting eigenvalue distribution of G_N is also the circular law.

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